COHOMOLOGY OVER FIBER PRODUCTS OF LOCAL RINGS

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ABSTRACT. Let S and T be local rings with common residue field k, let R be the fiber product $S \times_k T$, and let M be an S-module. The Poincaré series P_M^R of M has been expressed in terms of P_M^S, P_k^S and P_k^T by Kostrikin and Shafarevich, and by Dress and Krämer. Here, an explicit minimal resolution, as well as theorems on the structure of $\mathrm{Ext}_R(k,k)$ and $\mathrm{Ext}_R(M,k)$ are given that illuminate these equalities. Structure theorems for the cohomology modules of fiber products of modules are also given. As an application of these results, we compute the depth of cohomology modules over a fiber product.

INTRODUCTION

In homological investigations one often has information on properties of a module over a certain ring, and wants to extract information on its properties over a different ring. In this paper we consider the following situation: $S \to k \leftarrow T$ are surjective homomorphisms of rings, k is a field, R is the fiber product $S \times_k T$, and M an S-module. We further assume that S and T are either local rings with common residue field k, or connected graded k-algebras.

The starting point of this paper is the construction of an explicit minimal free resolution of M, viewed as an R-module, from minimal resolutions of M and k over S and k over T. This is carried out in section 2. The structure of the R-free resolution allows us to obtain precise information on the multiplicative structure of cohomology over R. Some of the results obtained in this work have been proved in the graded case by use of standard resolutions. However, no similar approach can be used in the local case.

The symbol $\sqcup$ denotes a coproduct, also known as a free product, of k-algebras.

Theorem A. *The canonical homomorphism of graded k-algebras*

$$\mathrm{Ext}_S(k,k) \sqcup \mathrm{Ext}_T(k,k) \to \mathrm{Ext}_{S \times_k T}(k,k)$$

defined by the universal property of coproducts of k-algebras is bijective. For every S-module M, the canonical homomorphism of graded left $\mathrm{Ext}_R(k,k)$-modules

$$\mathrm{Ext}_R(k,k) \otimes_{\mathrm{Ext}_S(k,k)} \mathrm{Ext}_S(M,k) \to \mathrm{Ext}_R(M,k)$$

defined by the multiplication map, is bijective.

These isomorphisms relate the Poincaré series $P_M^R(t)$ of M over R to $P_M^S(t)$, $P_k^S(t)$ and $P_k^T(t)$; this relationship was proved for $M = k$ by Kostrikin and Shafarevich [6], and by Dress and Krämer [3, Theorem 1] in the present setting. In [7], Polishchuk

and Positselski proved the preceding theorem, when S and T are connected k-algebras by using cobar constructions.

By combining Theorem A with an observation of Dress and Krämer concerning second syzygy modules over fiber products, we obtain the following corollary.

Corollary B. *Let L be an R-module. One then has $\Omega^2 L = M \oplus N$ where M and N are S and T-modules respectively, and an exact sequence of graded left $\mathcal{R}$-modules*

$$0 \to \left(\Sigma^{-2} \mathcal{R} \otimes_{\mathcal{S}} \operatorname{Ext}_S(M, k) \right) \oplus \left(\Sigma^{-2} \mathcal{R} \otimes_{\mathcal{T}} \operatorname{Ext}_T(N, k) \right) \to \mathcal{L} \to \mathcal{L}/\mathcal{L}^{\geq 2} \to 0$$

where $\mathcal{R} = \operatorname{Ext}_R(k, k)$, $\mathcal{S} = \operatorname{Ext}_S(k, k)$, $\mathcal{T} = \operatorname{Ext}_T(k, k)$ and $\mathcal{L} = \operatorname{Ext}_R(L, k)$.

Theorem A shows that $\operatorname{Ext}_-(k, k)$, as a functor in the ring argument, transforms products into coproducts. We show that $\operatorname{Ext}_R(-, k)$, as a functor from R-modules to $\operatorname{Ext}_R(k, k)$-modules, has a similar property. In the graded setting, this was shown by Polishchuk and Positselski [7]. Their methods do not extend to the local case, where even the equality of Poincaré series given is new.

Theorem C. *Let M, N, and V be S, T and k-modules respectively, so that we may define the R-module $M \times_V N$ as in Theorem 3.7. The exact sequence of R-modules*

$$0 \to M \times_V N \xrightarrow{\iota} M \times N \xrightarrow{\mu - \nu} V \to 0$$

induces an exact sequence of graded left $\operatorname{Ext}_R(k, k)$-modules

$$0 \to \operatorname{Ext}_R(V, k) \xrightarrow{\left(\begin{smallmatrix} \mu^* \\ -\nu^* \end{smallmatrix} \right)} \operatorname{Ext}_R(M, k) \oplus \operatorname{Ext}_R(N, k) \xrightarrow{\iota^*} \operatorname{Ext}_R(M \times_V N, k) \to 0.$$

In section 4 we study the depth of $\operatorname{Ext}_R(M, k)$ over $\operatorname{Ext}_R(k, k)$ for an R-module M. The notion of depth was used in [4] to study the homotopy Lie algebras of simply connected CW complexes, and of local rings. More recently, Avramov and Veliche [1] have shown that small depth of $\operatorname{Ext}_R(M, k)$ over $\operatorname{Ext}_R(k, k)$ is responsible for significant complications in the structure of the stable cohomology of M over R. For cohomology modules of R-modules, large depth is impossible.

Theorem D. *Let M be an R-module. Then*

$$\operatorname{depth}_{\operatorname{Ext}_R(k,k)} \operatorname{Ext}_R(M, k) \leq 1,$$

with equality if M is an S-module. In particular, one has $\operatorname{depth} \operatorname{Ext}_R(k, k) = 1$.

1. Graded Hilbert series and modules

In this section, we set notation for the entire article. Let k be a field.

1.1. A k-algebra A is *graded* if there is a decomposition of $A = \bigoplus_{i \in \mathbb{Z}} A_i$ as k-vector spaces, and for all $i, j \in \mathbb{Z}$, one has $A_i A_j \subseteq A_{i+j}$. We use both upper and lower indexed graded objects and adopt the notation $A^i = A_{-i}$. One says that A is *connected* if $A_0 = K$ and $A_i = 0$ for $i < 0$ (or equivalently, $A^i = 0$ for $i > 0$).

1.2. A left module M over a graded k-algebra A is *graded* if there is a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as k-vector spaces, and for all $i, j \in \mathbb{Z}$, one has $A_i M_j \subseteq M_{i+j}$.

1.3. Let M be a graded free K-module such that $M_i = 0$ for $i \ll 0$ and $\dim_k M_i$ is finite for all i. The Hilbert series of M is the formal Laurent series

$$M(t) = \sum_i \operatorname{rank}_k M_i t^i.$$

If $M(t)$ is defined, we say that M has a Hilbert series. Let $\mathbb{T}(M)$ denote the tensor algebra of M over A.

Lemma 1.4. *Let M and N be graded free A-modules with Hilbert series. Then one has an equality*

$$M \otimes_A N(t) = M(t)N(t).$$

If $M_i = 0$ for $i \leq 0$, then one also has

$$\mathbb{T}(M)(t) = \frac{1}{1 - M(t)}$$

Proof. The first claim is clear. For the second, since $M_i = 0$ for $i \leq 0$, $\mathbb{T}(M)$ has a Hilbert series. Furthermore, there is an isomorphism as graded A-modules $\mathbb{T}(M) \cong A \oplus (M \otimes_A \mathbb{T}(M))$. The desired result follows. $\square$

1.5. If A and B are graded connected k-algebras, the coproduct of A and B in this category is the free product of A and B, denoted $A \sqcup B$, and can be described as follows: A k-basis for $(A \sqcup B)_n$ consists of all elements of the form

$$a_1 \otimes b_2 \otimes \cdots \otimes a_l, \qquad a_1 \otimes b_2 \otimes \cdots \otimes b_m,$$
$$b_1 \otimes a_2 \otimes \cdots \otimes b_p \quad \text{and} \quad b_1 \otimes a_2 \otimes \cdots \otimes a_q$$

where a_i and b_j range over homogeneous bases of A and B respectively, and the total degree of an elementary tensor(given by summing the degrees of its terms) is n. Multiplication in $A \sqcup B$ is given by

$$(v \otimes \cdots \otimes x)(y \otimes \cdots \otimes w) = \begin{cases} v \otimes \cdots \otimes xy \otimes \cdots \otimes w & \text{for } x, y \in A \text{ or } x, y \in B \\ v \otimes \cdots \otimes x \otimes y \otimes \cdots \otimes w & \text{otherwise.} \end{cases}$$

We say that an element of the form $a \otimes \cdots \otimes b$ with $a \in A$ and $b \in B$ starts with A and ends with B.

1.6. Let M be a graded left A-module, then $(A \sqcup B) \otimes_A M$ is a graded left $A \sqcup B$-module. A k-basis for $((A \sqcup B) \otimes_A M)_n$ consists of all elements of the form

$$a_1 \otimes b_2 \otimes \cdots \otimes b_p \otimes m_{p+1} \quad \text{and} \quad b_1 \otimes a_2 \otimes \cdots \otimes b_q \otimes m_{q+1}$$

where a_i, b_j, and m_k range over homogeneous bases of A, B and M respectively, and the total degree of an elementary tensor is n. The action of $A \sqcup B$ is given by

$$(v \otimes \cdots \otimes x)(y \otimes \cdots \otimes w) = \begin{cases} v \otimes \cdots \otimes xy \otimes \cdots \otimes w & \text{for } x, y \in A \text{ or } x, y \in B \\ v \otimes \cdots \otimes xw & \text{for } x \in A \text{ and } y \in P \\ v \otimes \cdots \otimes x \otimes y \otimes \cdots \otimes w & \text{otherwise.} \end{cases}$$

Lemma 1.7. *Let A and B be graded conected k-algebras and M a graded left A-module. There is an equality of Hilbert series*

$$((A \sqcup B) \otimes_A M)(t) = \frac{M(t)B(t)}{A(t) + B(t) - A(t)B(t)}.$$

Proof. The basis given in 1.5 shows there is an isomorphism of k-vector spaces

$$A \sqcup B \cong B \otimes_k \mathbb{T}(A_+ \otimes_k B_+) \otimes_k A.$$

Tensoring over A with M on the right gives

$$(A \sqcup B) \otimes_A M \cong (B \otimes_k \mathbb{T}(A_+ \otimes_k B_+)) \otimes_k M.$$

A computation of Hilbert series gives the desired equality. $\square$

2. Resolutions over a fiber product

We consider a diagram of homomorphisms of rings

(2.1)
$$
\begin{array}{ccc}
S \times_k T & \xrightarrow{\;\tau\;} & T \\
{\scriptstyle\sigma}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi_T} \\
S & \xrightarrow[\;\pi_S\;]{} & k
\end{array}
$$

where π_S and π_T are surjective, and $S \times_k T$ is the fiber product:

$$ S \times_k T = \{(s,t) \in S \times T : \pi_S(s) = \pi_T(t)\}. $$

We set $\mathfrak{p} = \operatorname{Ker}\pi_S$, $\mathfrak{q} = \operatorname{Ker}\pi_T$, and $\mathfrak{m} = \operatorname{Ker}\pi_S\sigma = \operatorname{Ker}\pi_T\tau$. One then has $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$ and we identify $\mathfrak{p}$ and $\mathfrak{q}$ with subsets of R. Every S-module is considered an R-module via σ, and similarly for T-modules.

2.1. In the sequel we assume that we are in one of the following situations:
- S and T are commutative, noetherian, local rings with common residue field k, $\mathfrak{p}$ and $\mathfrak{q}$ the maximal ideals of S and T respectively, and M is a finitely generated S-module; or
- S and T are non-negatively graded, connected, degree-wise finite k-algebras, $\mathfrak{p} = S_+$, $\mathfrak{q} = T_+$, and M is a graded, bounded below, degree-wise finite S-module.

For a ring A as in 2.1, we say that a complex of free A-modules X is *minimal* if it satisfies $\partial(X) \subseteq \mathfrak{m}X$, where $\mathfrak{m}$ is the (homogeneous) maximal ideal of A. Note that every S-module M we consider has a minimal free resolution in which each free module is finitely generated.

If L is an A-module, the Poincaré series of L over A is the formal power series

$$ P_L^A(t) := \sum_i \dim_k \operatorname{Ext}_A^i(L,k)\, t^i. $$

Thus the coefficient of t^i is the rank of the ith free module in a minimal resolution of L over A, when one exists.

Definition 2.2. Let A be a ring and X a graded set, $\mathsf{X} = \bigsqcup_{n \geq 0} \mathsf{X}_n$. We let ${}^A\mathsf{X}$ denote the graded free left A-module with basis X_n in degree n, and set ${}^A\mathsf{X}_n = 0$ when $\mathsf{X}_n = \emptyset$. We call ${}^A\mathsf{X}$ a *graded based module* over A with basis X. Homomorphisms of based modules are identified with their matrices in the chosen bases.

For a based module ${}^A\mathsf{X}$, we identify $A \otimes_A {}^A\mathsf{X}$ and ${}^A\mathsf{X}$ by means of the canonical isomorphism. We use ${}^A[\mathsf{XY}]$ to denote the graded based A-module ${}^A\mathsf{X} \otimes_A {}^A\mathsf{Y}$ with graded basis $\mathsf{XY} = \bigsqcup_n [\mathsf{XY}]_n$, where $[\mathsf{XY}]_n$ is the set of symbols

$$ \{xy \mid x \in \mathsf{X}_i, y \in \mathsf{Y}_j, i + j = n\}. $$

Construction 2.3. Let M be an S-module. Let $P \to M$ and $E \to k$ be free resolutions of M, respectively k, over S, and let $T \to k$ be a free resolution of k over T such that $E_0 = S$ and $F_0 = T$. Choose bases P, E, and F of the graded modules P, E, and F over S, S and T, respectively, so that $\mathsf{E}_0 = \{1_S\}$ and $\mathsf{F}_0 = \{1_T\}$. Consider the elements of P, $\mathsf{E}_{\geq 1}$ and $\mathsf{F}_{\geq 1}$ as letters of an alphabet. The degree of a word in this alphabet is defined to be the sum of the degrees of its letters.

Let G be the set of all words of the form

$$ \{f_1 e_2 f_3 \cdots e_{2l-2} f_{2l-1} p_{2l}\} \quad \text{and} \quad \{e_1 f_2 e_3 \cdots e_{2l-1} f_{2l} p_{2l+1}\} $$

where e_i, f_i and p_i range over $\mathsf{E}_{\geq 1}, \mathsf{F}_{\geq 1}$ and P respectively, and $l \geq 0$. Form the free graded R-module $G = {}^R\mathsf{G}$.

Every word $w \in \mathsf{G}$ has the form xw' for some letter x and a (possibly empty) word w'. Assume that one has $\partial(E) \subseteq \mathfrak{p}E$, $\partial(P) \subseteq \mathfrak{p}P$, $\partial(F) \subseteq \mathfrak{q}F$, and set

$$\partial^G(w) = \begin{cases} \partial^P(x) & \text{for } x \in \mathsf{P} \\ \partial^E(x)w' & \text{for } x \in \mathsf{E} \\ \partial^F(x)w' & \text{for } x \in \mathsf{F}, \end{cases}$$

and extend ∂^G to a endomorphism of G by R-linearity. Set $\partial_i^G = \partial|_{G_i}$. We remark that a matrix φ with entries in $\mathfrak{p}$ defines a homomorphism ${}^S\varphi$ of free S-modules, as well as a homomorphism ${}^R\varphi$ of free R-modules.

Remark 2.4. The first few degrees of the complex G in Construction 2.3 looks as follows:
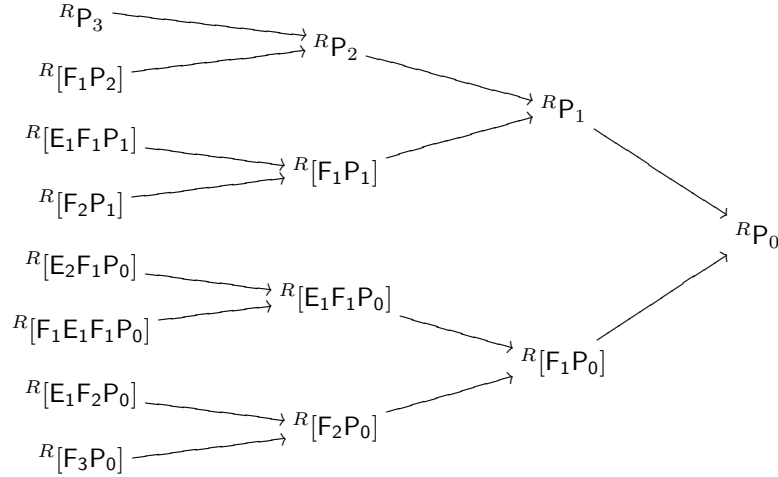
Note that each map in the diagram acts on the leftmost letter of a word.

Remark 2.5. Assume that we are in the graded situation of 2.1. If P, E and F are complexes of graded S, S and T-modules respectively, then G is a complex of graded R-modules; the internal degree of a word is the sum of the internal degrees of the letters in the word.

Theorem 2.6. *Assume that:*

- *M is an S-module with a minimal free resolution P,*
- *$S/\mathfrak{p}$ has a minimal free resolution E,*
- *$T/\mathfrak{q}$ has a minimal free resolution F.*

The maps of free modules ∂_i^G defined in Construction 2.3

$$G\colon \cdots \to G_i \xrightarrow{\partial_i^G} G_{i-1} \to \cdots \to G_1 \xrightarrow{\partial_1^G} G_0 \longrightarrow 0,$$

give a free resolution of the R-module M and satisfies $\partial^G(G) \subseteq \mathfrak{m}(G)$.

The following corollary was first obtained in [6] for $M = k$ and in [3, Theorem 1] in general.

Corollary 2.7. *There is an equality of Poincaré series:*

$$P_M^R(t) = \frac{P_M^S(t) P_k^T(t)}{P_k^S(t) + P_k^T(t) - P_k^S(t) P_k^T(t)}.$$

Proof. One may describe the basis G in Construction 2.3 as a basis of k-vector space ${}^k\mathsf{F} \otimes_R \mathbb{T}\left({}^k\mathsf{E}_{\geq 1} \otimes_k {}^k\mathsf{F}_{\geq 1}\right) \otimes_k {}^k\mathsf{P}$. Therefore, Lemma 1.4 gives

$$G(t) = \frac{P(t)F(t)}{1 - (E(t) - 1)(F(t) - 1)}.$$

The resolutions used in Construction 2.3 are minimial, so one has $G(t) = P_M^R(t)$, $E(t) = P_k^S(t)$, etc. Thus, the formula above gives the desired equality.   □

Proof of Theorem 2.6. To show that G is a complex, let w be a word of degree i, with $i \geq 2$. Suppose $w = xyw'$ where w' is a word, x is a letter of degree 1 and y is an arbitrary letter. For $x \in \mathsf{E}$ and $y \in \mathsf{F}$ one has

$$\partial^2(w) = \partial(\partial^E(x)yw') \in \partial(\mathfrak{p}yw') = \mathfrak{p}\partial(yw') = \mathfrak{p}\partial^F(y)w' \subseteq \mathfrak{pq}w' = 0.$$

The cases with $x \in \mathsf{F}_1$ and $y \in \mathsf{P}$, and with $x \in \mathsf{F}_1$ and $y \in \mathsf{E}$ are similar. If $w = xw'$ where x is a letter of degree greater than or equal to 2, then $\partial^2(w) = 0$, since ${}^R\mathsf{P}, {}^R\mathsf{E}$, and ${}^R\mathsf{F}$ are complexes of R-modules, and hence one has $\partial^2(x) = 0$.

Let ${}^R[\mathsf{E}_{\geq 2}\mathsf{G}]$ denote the R-linear span of words whose first letter is in E_i for some $i \geq 2$, see Definition 2.2. Let ${}^R[\mathsf{F}_1\mathsf{E}_{\geq 1}\mathsf{G}]$ denote the span of words starting with a letter from F_1, followed by a letter from E. Symbols such as ${}^R[\mathsf{F}_{\geq 2}\mathsf{G}]$, ${}^R[\mathsf{E}_1\mathsf{F}_{\geq 1}\mathsf{G}]$, etc. are defined similarly.

Let a be an element of G_i, with $i \geq 1$. It has a unique expression

$$a = (x + x') + (y + y') + (z + z') \quad \text{where}$$
$$x \in {}^R[\mathsf{E}_{\geq 2}\mathsf{G}], y \in {}^R[\mathsf{F}_{\geq 2}\mathsf{G}], z \in {}^R\mathsf{P},$$
$$x' \in {}^R[\mathsf{F}_1\mathsf{E}_{\geq 1}\mathsf{G}], y' \in {}^R[\mathsf{E}_1\mathsf{F}_{\geq 1}\mathsf{G}], z' \in {}^R[\mathsf{F}_1\mathsf{P}].$$

Notice that one has

$$\partial(x + x') \in {}^R[\mathsf{E}_{\geq 1}\mathsf{G}], \quad \partial(y + y') \in {}^R[\mathsf{F}_{\geq 1}\mathsf{G}], \quad \partial(z + z') \in {}^R\mathsf{P}.$$

Thus, $\partial(a) = 0$ implies that each one of $x + x', y + y'$ and $z + z'$ is a cycle. Next, we show each is a boundary, by giving details for $x + x'$; the other cases are similar.

Since one has $\mathfrak{p} \cap \mathfrak{q} = 0$ in R, and G_{i-1} is a free R-module, it follows that $\mathfrak{p}(G_{i-1}) \cap \mathfrak{q}(G_{i-1}) = 0$. Therefore, $\partial(x + x') = 0$ implies that x and x' are cycles as well. Let $l(w)$ denote the leftmost letter in the word w. We may express x according to the decomposition

$${}^R[\mathsf{E}_{\geq 2}\mathsf{G}]_i = \bigoplus_{\substack{2 \leq j \leq i \\ w \in \mathsf{G}_{i-j} \\ l(w) \in \mathsf{F}}} {}^R[\mathsf{E}_j w].$$

If w is basis element of degree $i - j$ with $l(w) \in F$ and $2 \leq j \leq i$, then one has $\partial({}^R[\mathsf{E}_j w]) \subseteq {}^R[\mathsf{E}_{j-1}w]$. Hence, each component of x in the decomposition above is a cycle. For similar reasons the components of x' in ${}^R[\mathsf{F}_1\mathsf{E}_{\geq 1}\mathsf{G}]$ are cycles.

Therefore, it is enough to show that every cycle of the form

$$x = \sum_{e \in \mathsf{E}_j} r_e ew \in {}^R[\mathsf{E_j}w] \quad \text{or}$$

$$x' = \sum_{f \in \mathsf{F}_1} \sum_{e \in \mathsf{E}_{j-1}} r_{fe} few \in {}^R[\mathsf{F_1E_{j-1}}w]$$

where w is a fixed word, and r_e, r_{ef} are in R, is a boundary. We give details for x, the other case is similar.

We first show that $r_e \in \mathfrak{m}$ for each $e \in \mathsf{E}_j$. Indeed, there is a commutative diagram of R-modules

(2.2)
$$\cdots \longrightarrow {}^R[\mathsf{E_3}w] \longrightarrow {}^R[\mathsf{E_2}w] \longrightarrow {}^R[\mathsf{E_1}w] \longrightarrow Rw \longrightarrow 0$$
$$\Big\downarrow{\widetilde{\sigma}_3} \qquad\quad \Big\downarrow{\widetilde{\sigma}_2} \qquad\quad \Big\downarrow{\widetilde{\sigma}_1} \qquad\quad \Big\downarrow{\widetilde{\sigma}_0}$$
$$\cdots \longrightarrow E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow 0$$

where the vertical maps send $\sum_{e \in \mathsf{E}_j} r_e ew$ to $\sum_{e \in \mathsf{E}_j} \sigma(r_e)e$. The image of x in E is a cycle, and hence a boundary, of E. As E is minimal, the claim follows.

Suppose $t = \partial(f)$ for $f \in \mathsf{F}_1$. Then $tew = \partial(few)$ is a boundary of G. As F is a resolution of k over T, the images of F_1 form a minimal generating set for $\mathfrak{q}$. Hence $\mathfrak{q}ew$ consists entirely of boundaries.

The claims above show it suffices to prove the theorem when the coefficients are in $\mathfrak{p}$. In diagram (2.2), we may also define a morphism of complexes $\widetilde{\gamma}: \mathfrak{p}E \to \mathfrak{p}{}^R[\mathsf{E}w]$ by sending $\sum_{e \in \mathsf{E}_j} s_e e$ to $\sum_{e \in \mathsf{E}_j} s_e ew$, viewing $s_e \in \mathfrak{p} \subset R$. Note that $\widetilde{\gamma}$ and $\widetilde{\sigma}\mid_{\mathfrak{p}{}^R[\mathsf{E}w]}$ are inverses of one another.

Suppose that $x \in \mathfrak{p}{}^R[\mathsf{E}w]$ is a cycle. Then $\widetilde{\sigma}(x)$ is a cycle in E. Hence there exists u so that $\partial^E(u) = \widetilde{\sigma}(x)$. Then one has

$$\partial(\widetilde{\gamma}(u)) = \widetilde{\gamma}(\partial^E(u)) = \widetilde{\gamma}(\widetilde{\sigma}(x)) = x. \qquad \square$$

Two special cases of the theorem are used in section 3.

Example 2.8. When $M = k$, we can take $\mathsf{P} = \mathsf{E}$. Let D be the resolution given by Theorem 2.6. Since $\mathsf{P}_0 = \{1\}$, we can also replace all basis elements of the form $w1$ with a basis element w of the same degree, and set $\partial^D(1_R) = 0$. Therefore, in low degrees, D has the form
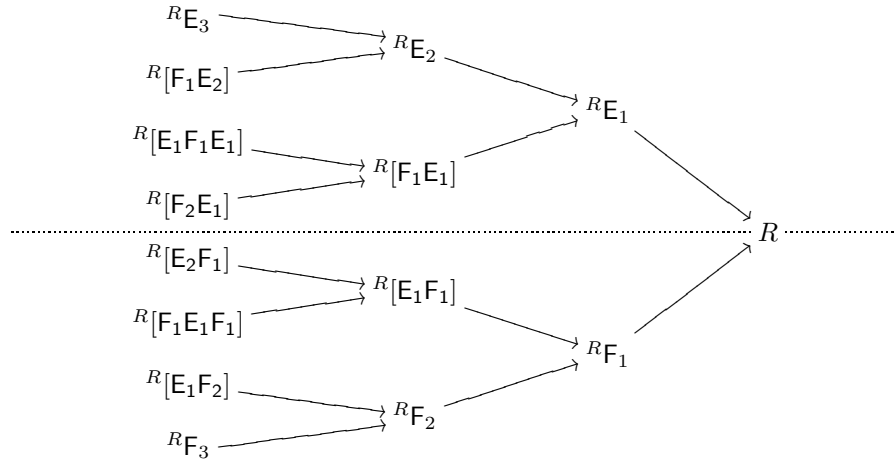
Example 2.9. When $M = T$, applying the theorem with the roles of S and T reversed, one has $P_i = 0$ for $i \neq 0$, and $\mathsf{P}_0 = \{1\}$. Let C be the resolution given by Theorem 2.6. Letting w denote the basis element $w1$ as above, we see that C is given by the top half of the diagram in Example 2.8.

3. The Yoneda algebra

Let $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ denote the Ext algebras of R, S, and T respectively. For an S-module M, we let $\mathcal{M}_S$ be the graded left $\mathcal{S}$-module $\mathrm{Ext}_S(M, k)$, and let $\mathcal{M}_R$ be the graded left $\mathcal{R}$-module $\mathrm{Ext}_R(M, k)$.

The functor $\mathrm{Ext}_-(k, k)$ applied to the diagram of homomorphisms of rings

$$
\begin{array}{ccc}
R & \xrightarrow{\tau} & T \\
\sigma \downarrow & & \downarrow \pi_T \\
S & \xrightarrow{\pi_S} & k
\end{array}
\qquad
\begin{array}{c}
\text{induces a diagram} \\
\text{of graded algebras}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{R} & \xleftarrow{\tau^*} & \mathcal{T} \\
\sigma^* \uparrow & & \uparrow \pi_S^* \\
\mathcal{S} & \xleftarrow{\pi_T^*} & k
\end{array}
$$

and hence defines a unique homomorphism of graded k-algebras

$$\phi : \mathcal{S} \sqcup \mathcal{T} \to \mathcal{R}$$

Theorem 3.1. *The homomorphism of connected k-algebras ϕ is an isomorphism.*

The homomorphism $\sigma : R \to S$ also induces a homomorphism of graded $\mathcal{S}$-modules $\sigma_M^* : \mathcal{M}_S \to \mathcal{M}_R$. Since σ_M^* is σ^*-equivariant, the formula $\xi \otimes \mu \mapsto \xi \cdot \sigma_M^*(\mu)$ defines a homomorphism of graded left $\mathcal{R}$-modules

$$\theta : \mathcal{R} \otimes_{\mathcal{S}} \mathcal{M}_S \to \mathcal{M}_R.$$

Theorem 3.2. *The homomorphism of graded left $\mathcal{R}$-modules θ is an isomorphism.*

In order to prove Theorems 3.1 and 3.2, we set up notation and describe the multiplication tables for $\mathcal{R}$ and $\mathcal{M}_R$.

Notation 3.3. In the notation of Construction 2.3, one sees that

$$\mathcal{M}_R \cong \mathrm{Hom}_R(G, k) \cong \mathrm{Hom}_R(G/\mathfrak{m}G, k) \cong \mathrm{Hom}_k(G/\mathfrak{m}G, k)$$

is a k-vector space. Let $\{\xi_w^R \mid w \in \mathsf{G}\}$ be the graded basis dual to the image of G in $G/\mathfrak{m}(G)$. Also, let $\{\xi_e^S \mid e \in \mathsf{E}\}$ be the graded basis dual to the basis given by the image of E in $E/\mathfrak{p}E$. We will abuse language and say that ξ_w^R starts (respectively ends) with a letter from E if the first (respectively last) letter of w is in E. Also, we will say that ξ_w^R has length n if w has length n.

Our first lemma concerns the image of words of length one.

Lemma 3.4. *For $e \in \mathsf{E}$, $f \in \mathsf{F}$, and $p \in \mathsf{P}$, one has*

$$\sigma^*(\xi_e^S) = \xi_e^R, \quad \tau^*(\xi_f^S) = \xi_f^R \quad \text{and} \quad \sigma_M^*(\xi_p^S) = \xi_p^R.$$

Proof. Let $\epsilon^R : D \to k$ and $\epsilon^S : E \to k$ be the augmentation maps. Set

$$D' = {}^R(\mathsf{D} \setminus \mathsf{E}) + \mathfrak{q}E \subseteq D.$$

The definition of ∂ shows that D' is a subcomplex. Also, since $R/\mathfrak{q} \cong S$ one has $D/D' = E$ as complexes of R-modules. Let ψ be canonical surjection $\psi : D \to D/D' = E$. Then one has $\epsilon^R = \epsilon^S \psi$, and hence $\sigma^*(\xi_e^S) := \psi \xi_e^S = \xi_e^R$, as desired. The other cases are similar. $\qquad\square$

Next we provide a partial multiplication table for the action of $\mathcal{R}$ on $\mathcal{M}_R$.

Lemma 3.5. *For $w \in \mathsf{D} \cup \mathsf{G}$ with starting letter $l(w)$ and x a letter in $\mathsf{E} \cup \mathsf{F}$, one has*

$$\xi_x^R \cdot \xi_w^R = \begin{cases} \xi_{fw}^R & \text{if } l(w) \in \mathsf{E} \cup \mathsf{P} \text{ and } x = f \in \mathsf{F} \\ \xi_{ew}^R & \text{if } l(w) \in \mathsf{F} \text{ and } x = e \in \mathsf{E}. \end{cases}$$

Proof. Suppose $w = ew' \in \mathsf{G}_i$, with $e \in \mathsf{E}$. Let f be an element in F. We define a chain map $\psi_w \in \operatorname{Hom}_R(G, D)_{-i}$ such that $\epsilon^R \psi_w = \xi_w^R$ as follows. Set

$$\mathsf{G}^w = \{x \in \mathsf{G} \mid x \notin (\mathsf{G}w \cup \mathsf{GE}_{\geq j+1}w')\},$$

where $\mathsf{G}w$ denotes elements of G that end in w (including w), and $\mathsf{GE}_{\geq j+1}w'$ denotes elements of G ending in a letter of $\mathsf{E}_{\geq j+1}$, followed by w'. Let $^R[\mathsf{G}^w]$ be the free R-module generated by G^w.

The definition of ∂ shows $^R[\mathsf{G}^w]$ is a subcomplex of G. Note that $G/^R[\mathsf{G}^w]$ is a complex of free R-modules with basis $\mathsf{G}w \cup \mathsf{GE}_{\geq j+1}w'$ and differential given by restricting ∂ to $^R[\mathsf{G}w]$ and $^R[\mathsf{GE}_{\geq j+1}w']$. If $v = v'w \in \mathsf{G}w$, define $\alpha'(v) = v'$, and extend α' by R-linearity to all of $^R[\mathsf{G}w]$. Then for $v = v'w \in \mathsf{G}w$, one has

$$\alpha'(\partial(v)) = \alpha'(\partial(v'w)) = \alpha'(\partial(v')w) = \partial(v') = \partial(\alpha'(v)).$$

Let B be the subcomplex of $G/^R[\mathsf{G}^w]$ spanned by $\mathsf{GE}_{\geq j+1}w' \cup \{w\}$, and let C be the resolution of T as an R-module given in Example 2.9. As C is acyclic and B is a free complex, we may define $\alpha'' \colon B \to D$ by lifting the map that sends $w \in B_i$ to $1 \in C_0$ and composing with the inclusion $C \to D$. Note that the words in the image of α'' end in letters from E. Since $\alpha'(w) = 1_R = \alpha''(w)$, we may define

$$\alpha(v) := \begin{cases} \alpha'(v) & v \in {}^R[\mathsf{G}w] \\ \alpha''(v) & v \in {}^R[\mathsf{GE}_{\geq j+1}w'] \end{cases}$$

Let ψ_w denote the composition $G \twoheadrightarrow G/^R[\mathsf{G}^w] \xrightarrow{\alpha} D$.

Clearly, one has $\epsilon^R \psi_w = \xi_w^R$, hence $\xi_f^R \cdot \xi_w^R = \xi_f^R \psi_w$. Let $v \in \mathsf{G}$ be a word. If $v \in \mathsf{G}^w$, then $\psi_w(v) = 0$. If $v \in \mathsf{G}w$, then write $v = v'w$. Then

$$\xi_f^R \psi_w(v) = \xi_f^R(v') = \begin{cases} 1 & \text{if } v' = f \\ 0 & \text{otherwise.} \end{cases}$$

If $v \in (\mathsf{GE}_{\geq j+1})w'$, then $\psi_w(v)$ is in the span of words with rightmost letters in E. Hence $\xi_f^R \psi_w(v) = 0$. Therefore

$$\xi_f^R \cdot \xi_w^R = \xi_f^R \psi_w(v) = \left\{ \begin{array}{ll} 1 & \text{if } v = fw \\ 0 & \text{otherwise.} \end{array} \right\} = \xi_{fw}^R(v).$$

The other cases are similar, and often easier. $\qquad\square$

Proof of Theorem 3.1. Under the hypothesis of the theorem, the k-algebras $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ are degree-wise finite. By definition, one has

$$\mathcal{R}(t) = P_k^R(t), \quad \mathcal{S}(t) = P_k^S(t) \quad \text{and} \quad \mathcal{T}(t) = P_k^T(t).$$

Lemma 1.7 and Corollary 2.7 yield $\mathcal{R}(t) = \mathcal{S} \sqcup \mathcal{T}(t)$. As ϕ is a homogeneous k-linear map, it suffices to show that it is surjective. Lemma 3.5 shows that

$$\{\xi_e^R \mid e \in \mathsf{E}\} \cup \{\xi_f^R \mid f \in \mathsf{F}\}$$

generates $\mathcal{R}$ as a k-algebra. Lemma 3.4 shows that these generators are in the image of ϕ. $\qquad\square$

Proof of Theorem 3.2. By Lemma 1.7 and Corollary 2.7 the Hilbert series of $\mathcal{R} \otimes_{\mathcal{S}} \mathcal{M}_S$ and $\mathcal{M}_R$ are equal, so it is enough to show that θ is surjective. By Lemma 3.5, $\mathcal{M}_R$ is generated as a left $\mathcal{R}$-module by $\{\xi_p^R \mid p \in \mathsf{P}\}$. By Lemma 3.4, $\theta(1 \otimes \xi_p^S) = \xi_p^R$, and hence θ is surjective. $\qquad\square$

Recall that a graded module M over a connected algebra A is said to be *Koszul* when $\operatorname{Ext}_A^i(M, k)_j = 0$ if $j \neq i$. A connected algebra A is Koszul if k is Koszul as an A-module. By Remark 2.5, the homomorphisms ϕ and θ preserve the internal gradings of $\mathcal{S} \sqcup \mathcal{T}$ and $\mathcal{R}$, giving the next corollary. The equivalence of the first two conditions was proved in [2].

Corollary 3.6. *The following conditions are equivalent.*
  1. *The algebra R is Koszul.*
  2. *The algebras S and T are Koszul.*
  3. *There exists an S-module M that is Koszul as a R-module.* $\qquad\square$

The functor $\operatorname{Ext}_R(-, k)$ has a property similar to the one given in Theorem 3.1.

Theorem 3.7. *Let M, N and V be S, T and k-modules respectively, such that there exist surjective π_S and π_T-equivariant homomorphisms $M \xrightarrow{\mu} V \xleftarrow{\nu} N$ with $\ker \mu = \mathfrak{p}M$ and $\ker \nu = \mathfrak{q}N$. The exact sequence of R-modules*

$$0 \to M \times_V N \xrightarrow{\iota} M \times N \xrightarrow{\mu - \nu} V \to 0$$

induces an exact sequence of graded left $\mathcal{R}$-modules

$$0 \to \mathcal{R} \otimes_k V^* \xrightarrow{(\mu^*, -\nu^*)} \mathcal{M}_R \times \mathcal{N}_R \xrightarrow{\iota^*} \mathcal{L} \to 0$$

where $\mathcal{L} = \operatorname{Ext}_R(M \times_V N, k)$, and $V^* = \operatorname{Hom}_k(V, k)$. In particular, there is an equality of Poincaré series

$$P_{M \times_V N}^R(t) + (\operatorname{rank}_k V) P_k^R(t) = P_M^R(t) + P_N^R(t).$$

Proof. The sequence of R-modules defining $M \times_V N$ induces an exact sequence of graded $\mathcal{R}$-modules

$$\Sigma^{-1}\mathcal{L} \to \mathcal{R} \otimes_k V^* \xrightarrow{(\mu^*, -\nu^*)} \mathcal{M}_R \times \mathcal{N}_R \xrightarrow{\iota^*} \mathcal{L} \to \Sigma(\mathcal{R} \otimes_k V^*).$$

Thus, we need to show that $(\mu^*, -\nu^*)$ is injective. Set $n = \operatorname{rank}_k V$. One has S-linear maps $S^n \to M \xrightarrow{\mu} V$ that induce homomorphisms of graded $\mathcal{R}$-modules $\mathcal{S} \otimes_k V^* \to \mathcal{M}_S \to k^n$. Tensoring with $\mathcal{R}$ over $\mathcal{S}$ on the left, one obtains

$$\mathcal{R} \otimes V^* \xrightarrow{\mu^*} \mathcal{M}_R \to \mathcal{R} \otimes_{\mathcal{S}} k^n$$

since $\mathcal{M}_R \cong \mathcal{R} \otimes_{\mathcal{S}} \mathcal{M}_S$. Under the isomorphism in Theorem 3.1, the kernel of this composition, and hence μ^*, is in the span of elements the elements of $\mathcal{R} \otimes_k V^*$ ending in $\mathsf{E}_{\geq 1}$. Similarly, one can show that the kernel of ν^* is contained the span of those elements of $\mathcal{R} \otimes_k V^*$ ending in $\mathsf{F}_{\geq 1}$. Hence $\operatorname{Ker}(\mu^*, -\nu^*) = \operatorname{Ker} \mu^* \cap \operatorname{Ker} \nu^* = 0$. $\quad\square$

4. Depth of cohomology modules

The notation and conventions given in sections 2 and 3 are still in force.

In order to describe the cohomology module for an arbitrary R-module L, we use an observation of Dress and Krämer in [3, Remark 3]. Recall that the syzygy $\Omega_1^R L$ of an R-module L is the kernel of a free cover $F \to L$; it is defined uniquely up to isomorphism; for $n \geq 2$ one sets $\Omega_n^R L = \Omega_1^R \Omega_{n-1}^R L$.

Proposition 4.1. *Let L be a left R-module. Then $\Omega_2^R(L) \cong M \oplus N$ where M is an S-module and N is a T-module.*

Proof. Recall that the maximal ideal of R is $\mathfrak{m} = \mathfrak{p} \oplus \mathfrak{q}$. Let $\varphi : A \to B$ be a minimal free presentation of L over R. One then has

$$
\begin{aligned}
\Omega_2^R(L) = \operatorname{Ker} \varphi &= \operatorname{Ker} \varphi \cap \mathfrak{m}A \\
&= \operatorname{Ker} \varphi \cap (\mathfrak{p}A \oplus \mathfrak{q}A) \\
&= (\operatorname{Ker} \varphi \cap \mathfrak{p}A) \oplus (\operatorname{Ker} \varphi \cap \mathfrak{q}A)
\end{aligned}
$$

To see the last equality, suppose that (x_1, x_2) in $\mathfrak{p}A \oplus \mathfrak{q}A$ satisfies $\varphi((x_1, x_2)) = 0$. Note that $\varphi((x_1, 0))$ is in $\varphi(\mathfrak{p}A) \subseteq \mathfrak{p}B$ and $\varphi((0, x_2))$ is in $\varphi(\mathfrak{q}A) \subseteq \mathfrak{q}B$. Also, $\mathfrak{p}B \cap \mathfrak{q}B = 0$, hence $\varphi((x_1, 0)) = 0 = \varphi((0, x_2))$. Taking $M = \operatorname{Ker} \varphi \cap \mathfrak{p}A$ and $N = \operatorname{Ker} \varphi \cap \mathfrak{q}A$ gives the desired result. $\square$

By putting together Theorem 3.2 and Proposition 4.1, we obtain a nearly complete description of the cohomology of arbitrary R-modules.

Corollary 4.2. *Set $\mathcal{L} = \operatorname{Ext}_R(L, k)$, $\mathcal{M}_S = \operatorname{Ext}_S(M, k)$, and $\mathcal{N}_T = \operatorname{Ext}_T(N, k)$. There is then an exact sequence of graded left $\mathcal{R}$-modules*

$$
0 \to (\Sigma^{-2}\mathcal{R} \otimes_{\mathcal{S}} \mathcal{M}_S) \oplus (\Sigma^{-2}\mathcal{R} \otimes_{\mathcal{T}} \mathcal{N}_T) \to \mathcal{L} \to \mathcal{L}/\mathcal{L}^{\geq 2} \to 0
$$

Corollary 4.2 allows us to compute the depth of the cohomology module of an R-module. For uses of this invariant, see [4] or [1].

Definition 4.3. For a graded connected k-algebra $\mathcal{E}$ and a graded left $\mathcal{E}$-module $\mathcal{M}$, one defines the depth of $\mathcal{M}$ over $\mathcal{E}$ by means of the formula

$$
\operatorname{depth}_{\mathcal{E}} \mathcal{M} = \inf\{n \in \mathbb{N} \mid \operatorname{Ext}_{\mathcal{E}}^n(k, \mathcal{M}) \neq 0\}.
$$

For $M = k$, the following theorem recovers [5, Example 36.(e).2]. The idea to use the resolution of $\mathcal{S} \sqcup \mathcal{T}$ constructed from those of $\mathcal{S}$ and $\mathcal{T}$ is taken from there. Note that for each i, $\operatorname{Ext}_{\mathcal{R}}^i(k, \mathcal{M}_R)$ is a graded abelian group, with the internal grading given by the cohomological grading on $\mathcal{M}_R$. We will denote its jth graded piece by $\operatorname{Ext}_{\mathcal{R}}^i(k, \mathcal{M}_R)^j$.

More precisely, if $\mathcal{F}_i$ is the ith free module in a graded free resolution of k over $\mathcal{R}$, then $\operatorname{Ext}_{\mathcal{R}}^i(k, \mathcal{M}_R)$ carries the induced grading of $\operatorname{Hom}_{\mathcal{R}}(\mathcal{F}_i, \mathcal{M}_R)$. A homomorphism $\varphi \colon \mathcal{F}_i \to \mathcal{M}_R$ has degree j if it satisfies $\varphi(\mathcal{F}_i^n) \subseteq \mathcal{M}_R^{n-j}$ for all $n \in \mathbb{Z}$.

Theorem 4.4. *If M is a finitely generated non-zero S-module, and neither π_S nor π_T is an isomorphism. Then one has $\operatorname{depth} \mathcal{M}_R = 1$.*

More precisely, for each $j \geq 2$, one has

$$
\operatorname{Ext}_{\mathcal{R}}^1(k, \mathcal{M}_R)^j \oplus \operatorname{Ext}_{\mathcal{R}}^1(k, \mathcal{M}_R)^{j+1} \neq 0,
$$

unless S and T both have global dimension 1 and M is free, in which case

$$
\operatorname{Ext}_{\mathcal{R}}^1(k, \mathcal{M}_R)^1 \neq 0.
$$

Proof. As S and T are not fields, one can find $\varsigma \in \mathcal{S}^1 \setminus \{0\}$ and $\vartheta \in \mathcal{T}^1 \setminus \{0\}$. We fix these elements for the remainder of the proof. We first show that for each $\mu \in \mathcal{M}_R^i$, there is an element $\xi \in \mathcal{R}^+$ so that $\xi\mu \neq 0$. By Theorem 3.2 and 1.6, we may arrange the terms in μ so that $\mu = \alpha + \beta + \gamma$, where the terms in α start with a letter from E, the terms in β start with F and the terms in γ start with 1. If $\gamma \neq 0$, then the terms in $\vartheta\gamma$ start with a letter of F_1, and the length of each of

its terms is 2. But $\vartheta\alpha$ has terms of length greater than or equal to 3, and $\vartheta\beta$ has terms with the degree of their leading letters greater than 1. Therefore, no term in $\vartheta\mu$ can cancel $\vartheta\gamma$, so $\vartheta\mu \neq 0$. One can similarly argue that if α or β are nonzero, then $\vartheta\alpha$ or $\varsigma\beta$ are nonzero, respectively.

Choose free resolutions of k over $\mathcal{S}$ and $\mathcal{T}$ and write them in the form

$$\cdots \xrightarrow{\partial} \mathcal{S} \otimes_k V(2) \xrightarrow{\partial} \mathcal{S} \otimes_k V(1) \xrightarrow{\partial} \mathcal{S}$$

$$\cdots \xrightarrow{\partial} \mathcal{T} \otimes_k W(2) \xrightarrow{\partial} \mathcal{T} \otimes_k W(1) \xrightarrow{\partial} \mathcal{T}$$

with graded k-vector spaces $V(i)$ and $W(i)$. One then has a free resolution

(4.1) $$\cdots \xrightarrow{\partial} \mathcal{R} \otimes_k (V(2) \oplus W(2)) \xrightarrow{\partial} \mathcal{R} \otimes_k (V(1) \oplus W(1)) \xrightarrow{\partial} \mathcal{R}$$

of k over $\mathcal{R}$, where the differentials are given by restricting those in the resolutions of k over $\mathcal{S}$ and $\mathcal{T}$, see [5, Example 36.e.2].

For any pair of elements $\alpha, \beta \in \mathcal{M}_R^i$, define an $\mathcal{R}$-linear map

$$\phi_{\alpha,\beta} \colon \mathcal{R} \otimes_k (V(1) \oplus W(1)) \to \mathcal{M}_R$$
$$(v, w) \mapsto (\partial(v)\alpha, \partial(w)\beta)$$

Then $\phi_{\alpha,\beta}$ is a 1-cocycle and hence defines a class in $\mathrm{Ext}_{\mathcal{R}}(k, \mathcal{M}_R)$. If there exists $v \in V(1)$ or $w \in W(1)$ so that $(\partial v)\beta \neq 0$ or $(\partial w)\alpha \neq 0$, then $\phi_{\alpha,\beta}$ represents a nonzero cohomology class.

As M is nonzero, there exists $\mu \in \mathcal{M}_S^0 \setminus \{0\}$. If M is not free over S, then there also exists $\mu' \in \mathcal{M}_S^1 \setminus \{0\}$. If $\mathrm{gldim}\, S \geq 2$ (respectively $\mathrm{gldim}\, T \geq 2$), then there exists $\varsigma' \in \mathcal{S}^2 \setminus \{0\}$ (respectively $\vartheta' \in \mathcal{T}^2 \setminus \{0\}$). For each $j \geq 1$, define

$$\alpha_j = \begin{cases} (\vartheta\varsigma)^{j-1}\vartheta\mu & \text{if } M \text{ is not free} \\ (\vartheta\varsigma)^{j-1}\vartheta'\mu & \text{if } \mathrm{gldim}\, T \geq 2 \\ (\vartheta\varsigma)^j\vartheta\mu & \text{if } \mathrm{gldim}\, S \geq 2 \end{cases}$$

$$\beta_i = \begin{cases} (\varsigma\vartheta)^{j-1}\mu' & \text{if } M \text{ is not free} \\ (\varsigma\vartheta)^j\mu & \text{if } \mathrm{gldim}\, T \geq 2 \\ (\varsigma\vartheta)^{j-1}\varsigma'\vartheta\mu & \text{if } \mathrm{gldim}\, S \geq 2 \end{cases}$$

Then for each $j \geq 1$, ϕ_{α_j,β_j} defines a distinct nonzero class of $\mathrm{Ext}_{\mathcal{R}}^1(k, \mathcal{M}_R)$, with an internal degree of ϕ_{α_j,β_j} either $-2j+1$, $-2j$, or $-2j-1$ if M is not free, $\mathrm{gldim}\, T \geq 2$ or $\mathrm{gldim}\, S \geq 2$, respectively.

When S and T have global dimension one, the graded free resolution (4.1) is

$$0 \longrightarrow \mathcal{R} \otimes_k (V(1) \oplus W(1)) \xrightarrow{\partial} \mathcal{R}$$

Application of $\mathrm{Hom}_{\mathcal{R}}(-, \mathcal{M}_R)$ gives a short exact sequence:

$$0 \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{M}_R) \xrightarrow{\partial^*} \mathrm{Hom}_{\mathcal{R}}(\mathcal{R} \otimes_k (V(1) \oplus W(1)), \mathcal{M}_R) \to \mathrm{Ext}_{\mathcal{R}}^1(k, \mathcal{M}_R) \to 0$$

Since $\mathcal{S}$ and $\mathcal{T}$ are connected k-algebras, the graded bases $V(1)$ and $W(1)$ start in degree one, and hence $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R} \otimes_k (V(1) \oplus W(1)), \mathcal{M}_R)^1 \neq 0$. Furthermore, this component is not in the image of ∂^* since $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{M}_R)$ starts in degree zero and ∂^* is homogeneous. Hence $\mathrm{Ext}_{\mathcal{R}}^1(k, \mathcal{M}_R)^1 \neq 0$. $\square$

Using the short exact sequence given in Corollary 4.2, one also has the following:

Corollary 4.5. *For each non-zero finitely generated R-module L, one has*

$$\operatorname{depth}_{\mathcal{R}} \mathcal{L} \leq 1.$$

Proof. If $\operatorname{pd}_R L$ is finite, then $\operatorname{rank}_k \mathcal{L}$ is finite, hence $\operatorname{depth}_{\mathcal{R}} \mathcal{L} = 0$. If $\operatorname{pd}_R L = \infty$, then $\Omega_R^2(L) \neq 0$, and by Proposition 4.1, we have $\Omega_R^2(L) = M \oplus N$ for some S-module M and some T-module N, with M or N non-zero.

Suppose that either $\operatorname{gldim} S \geq 2$ or $\operatorname{gldim} T \geq 2$, and set $\mathcal{X} := \operatorname{Ext}_R(M \oplus N, k)$. Then $\operatorname{rank}_k \operatorname{Ext}^1_{\mathcal{R}}(k, \mathcal{X}) = \infty$, by Theorem 4.4. Then by Corollary 4.2, together with Theorem 4.4, there is an exact sequence

$$\operatorname{Hom}_{\mathcal{R}}(k, \mathcal{L}/\mathcal{L}^{\geq 2}) \xrightarrow{\varphi} \operatorname{Ext}^1_{\mathcal{R}}(k, \mathcal{X}) \xrightarrow{\psi} \operatorname{Ext}^1_{\mathcal{R}}(k, \mathcal{L}).$$

Since $\operatorname{rank}_k \operatorname{Hom}_{\mathcal{R}}(k, \mathcal{L}/\mathcal{L}^{\geq 2})$ is finite, φ is not surjective, and hence ψ is nonzero.

If both S and T have global dimension one, then one has $\operatorname{Ext}^1_{\mathcal{R}}(k, \mathcal{X})^1 \neq 0$ by Theorem 4.4. Since φ is a homogeneous homomorphism and $\operatorname{Hom}_{\mathcal{R}}(k, \mathcal{L}/\mathcal{L}^{\geq 2})$ is concentrated in nonnegative degrees, φ is not surjective, and hence ψ is nonzero. $\square$

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